Downsampling of Bounded Bandlimited Signals and the Bandlimited Interpolation: Analytic Properties and Computability

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***Abstract*—Downsampling and the computation of the bandlimited interpolation of discrete-time signals are two important concepts in signal processing. In this paper we analyze the downsampling operation regarding its impact on the existence and computability of the bounded bandlimited interpolation. We assume that the discrete-time signal is obtained by downsampling the samples of a bounded bandlimited signal that vanishes at infinity, and we study two problems. First, we investigate the existence of the bounded bandlimited interpolation for such discrete-time signals from a signal theoretic perspective and show that there exist signals for which the bounded bandlimited interpolation does not exist. Second, we analyze the algorithmic generation of the bounded bandlimited interpolation, using the concept of Turing computability. Turing computability models what is theoretically implementable on a digital computer. Interestingly, it turns out that even if the bounded bandlimited interpolation exists analytically, it is not always computable, which implies that there exists no algorithm on a digital computer that can always compute it. Computability is important in order that the approximation error be controlled. If a signal is not computable, we cannot ascertain whether the computed signal is sufficiently close to the true signal, i.e., we cannot verify every approximation accuracy.**

***Index Terms*—bandlimited signal, downsampling, bandlimited interpolation, Turing computability**

I. Introduction

SAMPLING of analog signals is one of the basic operations in signal processing and is of fundamental importance. The Shannon sampling theorem provides the theoretical foundation for the sampling of bandlimited signals with finite energy and their error-free recovery from the samples [2]. Since Shannon's paper from 1949, numerous publications have extended this result [3]–[8] in different directions. Among them are sampling theorems for other and larger signal spaces and for new basis representations [9]–[12]. In addition to signal processing, the sampling theorem has important applications in other disciplines, such as physics [13]. For a historical treatment, see, for example, [14]–[17].

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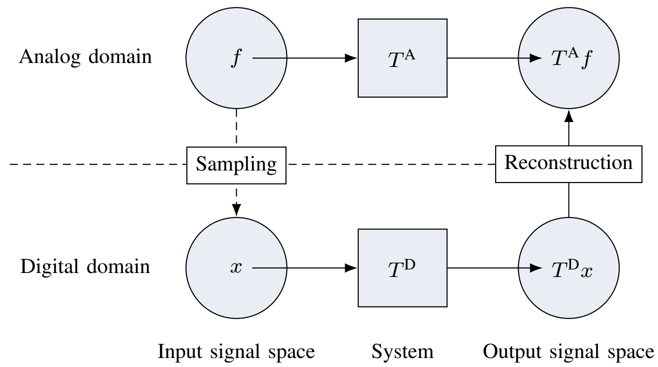

Fig. 1. Processing of analog signals in the digital domain.

Downsampling or decimation, which is the process of reducing the sampling rate of a discrete-time signal by removing samples, is a central operation in digital signal processing that is used in many applications, for example in filter banks [18], [19], image processing [20]–[22], and communication systems [23], [24]. In this work we consider only one-dimensional downsampling. If we downsample a signal $\{x_k\}_{k\in\mathbb{Z}}$ by a factor of two, we only keep the samples $\{x_{2k}\}_{k\in\mathbb{Z}}$, and the downsampled signal is given by $\{x_k^{\text{down}}\}_{k\in\mathbb{Z}} = \{x_{2k}\}_{k\in\mathbb{Z}}$. Often, the discrete-time signal is obtained by sampling a bandlimited continuous-time signal. Since we do not consider quantization in this paper, we also call a discrete-time signal a digital signal.

Digitization, i.e., the transition from the continuous-time domain to the discrete-time domain is the basis of today's digital transformation, where the key idea is to perform all signal processing operations in the digital domain. The original idea, which is illustrated in Fig. 1, is based on the Shannon sampling theorem. Instead of processing an analog signal f in the analog domain using an analog system T^{A} to produce the desired output signal $g = T^{\mathrm{A}} f$, the signal f is first converted into a digital signal x which is then processed by a digital system T^{D}, resulting in a digital output signal $y = T^{\mathrm{D}} x$. Finally, the digital signal y is converted into an analog signal $\tilde{g}$ by means of a reconstruction process. If $\tilde{g} = g$ then the analog signal processing task has been successfully transferred into a digital signal processing task, according to the above procedure.

Nowadays, digital signal processing is an independent discipline in signal processing, and often, the signals that are being processed are already created in the digital domain and do not

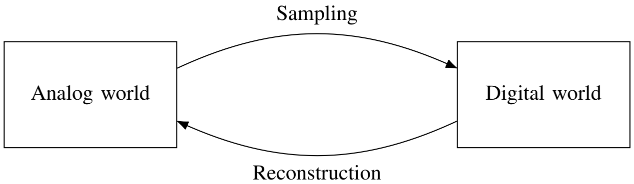

Fig. 2. Connection of the analog and the digital domain.

stem from analog signals. However, in many situations such as in digital communications, at the end of the processing the discrete-time signals are still transformed into the continuous-time domain, and this transformation is done by computing the bandlimited interpolation of the discrete-time signal. Hence, the questions of whether the bandlimited interpolation exists and whether it can be generated algorithmically are of high practical relevance.

For bandlimited signals f with finite energy and no frequencies larger than 2π, we know from Shannon's sampling theorem that we can reconstruct the continuous-time signal f from its samples $\{f(k/2)\}_{k\in\mathbb{Z}}$. Thus, by sampling and interpolation with the Shannon sampling series, we have a correspondence between discrete-time and continuous-time signals, as illustrated in Fig. 2. In this paper we study the impact of downsampling on this correspondence. Without loss of generality, we use bandlimited signals having a bandwidth of π or 2π. Then sampling at the Nyquist rate corresponds to sampling on the grids $\{k\}_{k\in\mathbb{Z}}$ and $\{k/2\}_{k\in\mathbb{Z}}$, respectively. However, this is no restriction, other bandwidths can be obtained by a simple scaling of the signals.

Assume that we have a bounded bandlimited signal f with bandwidth 2π that we sample at the Nyquist rate. Then the discrete-time signal is given by $\{x_k\}_{k\in\mathbb{Z}} = \{f(k/2)\}_{k\in\mathbb{Z}}$. In the next step we downsample this sequence by a factor of two, which gives us the downsampled signal $\{x_k^{\text{down}}\}_{k\in\mathbb{Z}} = \{f(k)\}_{k\in\mathbb{Z}}$. Now the question is whether for this downsampled discrete-time signal $\{x_k^{\text{down}}\}_{k\in\mathbb{Z}}$ there exist any problems in finding a bounded continuous-time signal f_π with bandwidth π that interpolates the downsampled signal $\{x_k^{\text{down}}\}_{k\in\mathbb{Z}}$, i.e., satisfies $f_\pi(k) = x_k^{\text{down}}, k \in \mathbb{Z}$. Such a signal f_π is known as the bounded bandlimited interpolation and has many applications, e.g., in communications and image processing [25]–[28].

We study two aspects of this question. First, we look at this problem from a signal theoretic perspective and ask if f_π exists as a mathematical object. This is clearly a necessary condition for any practical application. In general, downsampling as a signal processing operation is not given much attention in theoretical analyses, because it is assumed that this procedure does not create any fundamental problems. In many signal processing books, the bandlimited interpolation, i.e., the continuous-time signal that corresponds to the downsampled sequence, is formally obtained by using a convolution theorem and distribution theory. See, for example, [25, p. 52, p. 162], [26, p. 144]. However, for signal spaces other than the space of bandlimited signals with finite energy, it is a priori not clear whether those manipulations and expressions are well-defined, even when they are treated in the sense of distributions [29].

Second, we analyze a more subtle problem that has not gotten much attention in the signal processing community so far, which is, nevertheless, equally important for a practical implementation: the question of computability. Even if f_π exists as a signal in a mathematical sense, we need to be able to compute f_π from the discrete-time signal $\{x_k^{\text{down}}\}_{k\in\mathbb{Z}}$. That is, we need an algorithm that can approximate f_π from $\{x_k^{\text{down}}\}_{k\in\mathbb{Z}}$ in a finite number of steps and with assured precision. The theoretical concept that we employ to study this second question is Turing computability. Turing computability is a standard model for computing that idealizes our digital computers. A Turing machine has no limitations in terms of memory or computing time, and hence provides a theoretical model that describes the fundamental limits of any practically realizable digital computer.

The answers to both questions clearly depend on the properties of the involved signals. For example, if f is a computable 2π-bandlimited signal with finite energy, then the discrete-time signal $\{x_k\}_{k\in\mathbb{Z}} = f(k/2)$ and the downsampled signal $\{x_k^{\text{down}}\}_{k\in\mathbb{Z}} = f(k)$ both have finite energy and are computable. The bounded bandlimited interpolation f_π exists and can be obtained by means of the Shannon sampling series

$$f_\pi(t) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}. \tag{1}$$

In this case f_π is also computable, i.e., we can find an algorithm that for any given precision goal, approximates f_π. However, for other signal spaces, this is not necessarily true. In this paper we consider the space of bounded bandlimited signals that vanish at infinity.

After introducing some notation in Section II, we introduce the concept of computability in Section III and give some facts about downsampling and interpolation in Section IV. In Section V we analyze the first question, i.e., the existence of the bounded bandlimited interpolation. The second question, i.e., the computability of the bounded bandlimited interpolation, is treated in Section VI, followed by a discussion of the results in Section VII. The proofs of the theorems are given in Section VIII. In Section IX we discuss the size of certain sets of signals with problematic behavior, and in Section X we construct a non-computable number and discuss the approximation in terms of computable Cauchy sequences. The paper is concluded with a discussion in Section XI.

II. Notation

By c_0 we denote the set of all sequences that vanish at infinity, and by $C_0^\infty[0,1]$, the space of all functions that have continuous derivatives of all orders and are zero outside $[0,1]$. For $\Omega \subset \mathbb{R}$, let $L^p(\Omega)$, $1 \leq p < \infty$, be the space of all measurable pth-power Lebesgue integrable functions on Ω, with the usual norm $\|\cdot\|_p$, and $L^\infty(\Omega)$ the space of all functions for which the essential supremum norm $\|\cdot\|_\infty$ is finite. The Bernstein space $\mathcal{B}_\sigma^p$, $\sigma > 0$, $1 \leq p \leq \infty$, consists of all functions of exponential type at most σ, whose restriction to the real line is in $L^p(\mathbb{R})$ [8, p. 49]. The norm for $\mathcal{B}_\sigma^p$ is given by the L^p-norm on the real line. A function in $\mathcal{B}_\sigma^p$ is called bandlimited to σ. $\mathcal{B}_{\sigma,0}^\infty$ denotes the space of all functions in $\mathcal{B}_\sigma^\infty$ that vanish at infinity. By $\mathcal{PW}_\sigma^p$, $1 \le p \le \infty$, we denote the Paley–Wiener space of functions f with a representation $f(z) = 1/(2\pi) \int_{-\sigma}^{\sigma} g(\omega)\, \mathrm{e}^{iz\omega} \, \mathrm{d}\omega$, $z \in \mathbb{C}$, for some $g \in L^p[-\sigma, \sigma]$. If $f \in \mathcal{PW}_\sigma^p$, then $g(\omega) = \hat{f}(\omega)$. The norm for $\mathcal{PW}_\sigma^p$ is given by $\|f\|_{\mathcal{PW}_\sigma^p} = (1/(2\pi) \int_{-\sigma}^{\sigma} |\hat{f}(\omega)|^p \, \mathrm{d}\omega)^{1/p}$. $\mathcal{PW}_\sigma^2$ is the frequently used space of bandlimited functions with bandwidth σ and finite energy.

Distributions are continuous linear functionals on a space of test functions. $\mathcal{D}$ is the space of all test functions $\phi : \mathbb{R} \to \mathbb{C}$ that have continuous derivatives of all orders and are zero outside some finite interval. $\mathcal{D}'$ denotes the dual space of $\mathcal{D}$, i.e., the space of all distributions that can be defined on $\mathcal{D}$. For locally integrable functions g we can define the linear functional

$$\phi \mapsto \int_{-\infty}^{\infty} g(t)\phi(t) \, \mathrm{d}t \tag{2}$$

on the space $\mathcal{D}$. It can be proven that this functional is continuous and thus defines a distribution [30]. Distributions of the type (2) are called regular distributions. A sequence of distributions $\{f_k\}_{k\in\mathbb{N}}$ in $\mathcal{D}'$ is said to converge in $\mathcal{D}'$, if for every $\phi \in \mathcal{D}$ the sequence of numbers $\{f_k\phi\}_{k\in\mathbb{N}}$ converges. Thus, a sequence of regular distributions, which is induced by a sequence of functions $\{g_k\}_{k\in\mathbb{N}}$ according to (2), converges in $\mathcal{D}'$, if for every $\phi \in \mathcal{D}$ the sequence of numbers $\{\int_{-\infty}^{\infty} g_k(t)\phi(t) \, \mathrm{d}t\}_{k\in\mathbb{N}}$ converges.

III. Computability

The theory of computability is a well-established field in computer sciences [31]–[35]. However, since computability is not widely known in the signal processing community, we describe some of the key concepts in this section. For a more detailed treatment of the topic, see for example [33]–[36].

In order to study the question of computability, we employ the concept of Turing computability. A Turing machine is an abstract device that manipulates symbols on a strip of tape according to certain rules [31]–[33], [35]. Although the concept is very simple, a Turing machine is capable of simulating any given algorithm. Turing machines have no limitations in terms of memory or computing time, and hence provide a theoretical model that describes the fundamental limits of any practically realizable digital computer. Moreover, Turing machines are equivalent to other concepts of computability, such as those defined by general recursive functions, Minksy register machines, and λ-calculus.

It is important to distinguish Turing computability from complexity theory, another major topic in computer science. Complexity theory, which is also relevant for signal processing, deals with the question of how efficiently a problem can be solved, and analyzes how the computation time of a given algorithm scales with the size of the input data. Thus, the goal of complexity theory is different from the goal in Turing computability, where the fundamental limits of computability are explored without considering complexity issues. It is clear that the study of the complexity of a problem requires that the problem be algorithmically solvable.

Further, complexity theory operates in a discrete and finite setting. However, the modeling of many real world problems involves continuous infinite signals, e.g., bandlimited signals that have an infinite duration. Thus, in order to apply complexity theory on such "continuous problems", it is necessary that the continuous signals be approximated by discrete and finite signals in a controlled way.

Questions of complexity have been studied in signal processing for a long time. As for computability, this is not the case. It seems to be a folklore result in signal processing that an increase in computational power automatically leads to a better approximation of the continuous problem, where we implicitly control the approximation error between the computed signal and the true solution. That this is not the case for downsampling and the bandlimited interpolation will be the result of Section VI.

Alan Turing introduced the concept of a computable real number in [31], [32]. A sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ is called a computable sequence if there exist recursive functions a, b, s from $\mathbb{N}$ to $\mathbb{N}$ such that $b(n) \neq 0$ for all $n \in \mathbb{N}$ and

$$r_n = (-1)^{s(n)} \frac{a(n)}{b(n)}, \qquad n \in \mathbb{N}.$$

A recursive function is a function mapping natural numbers into natural numbers, which is built of simple computable functions and recursions [37]. Recursive functions are computable by a Turing machine. A real number x is said to be computable if there exists a computable sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ and a recursive function $\xi \colon \mathbb{N} \to \mathbb{N}$ such that for all $N \in \mathbb{N}$ we have $|x - r_n| \le 2^{-N}$ for all $n \ge \xi(N)$. By $\mathbb{R}_c$ we denote the set of computable real numbers and by $\mathbb{C}_c = \mathbb{R}_c + i\mathbb{R}_c$ the set of computable complex numbers. $\mathbb{R}_c$ is a field, i.e., finite sums, differences, products, and quotients of computable numbers are computable. Note that commonly-used constants like e and π are computable. A non-computable real number was, for example, constructed in [38].

There are several—not equivalent—definitions of computable functions, most notably, Turing computable functions, Markov computable functions, and Banach–Mazur computable functions [36]. An example of a non-computable function was given in [39]. A function that is computable with respect to any of the above definitions has the property that it maps computable numbers into computable numbers. This property is therefore a necessary condition for computability. Usual functions like sin, sinc, log, and exp are computable, and finite sums of computable functions are computable [34].

We call a function f elementary computable if there exists a natural number N and a sequence of computable numbers $\{\alpha_k\}_{k=-N}^{N}$, such that

$$f(t) = \sum_{k=-N}^{N} \alpha_k \frac{\sin(\pi(t-k))}{\pi(t-k)}. \tag{3}$$

Note that every elementary computable function f is a finite sum of computable functions and hence computable. As a consequence, for every $t \in \mathbb{R}_c$ the number $f(t)$ is computable. Further, the sum of finitely many elementary computable functions is computable, as well as the product of an elementary computable function with a computable number $\lambda \in \mathbb{C}_c$. Hence, the set of elementary computable functions is closed

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with respect to the operations addition and multiplication with a scalar. Further, as we will show next, for every elementary computable function f, the norm $\|f\|_{\mathcal{B}_{\pi,0}^\infty}$ is computable. Let f be an elementary computable function, having the shape (3). We have

$$\|f\|_{\mathcal{B}_{\pi,0}^\infty} \leq \max_{|k|\leq N}|\alpha_k| \cdot \max_{t\in\mathbb{R}} \sum_{k=-N}^{N} \left|\frac{\sin(\pi(t-k))}{\pi(t-k)}\right| \leq \max_{|k|\leq N}|\alpha_k| \left(5 + \frac{4}{\pi} + \frac{4}{\pi}\log(N)\right), \quad (4)$$

where the second inequality follows from a calculation that is similar to (20). Let $M > N+1$. A basic but lengthy calculation shows that

$$|f(t)| \leq \frac{1}{\pi} \max_{|k|\leq N}|\alpha_k| \log\left(\frac{M+N}{M-N-1}\right)$$

for all $|t| \geq M$. Let M_1 be the smallest natural number such that $M_1 > N+1$ and

$$\log\left(\frac{M+N}{M-N-1}\right) < \frac{\pi}{2}.$$

Then we have

$$|f(t)| < \frac{1}{2} \max_{|k|\leq N}|\alpha_k|$$

for all $|t| \geq M_1$. Since

$$\|f\|_{\mathcal{B}_{\pi,0}^\infty} \geq \max_{|k|\leq N}|\alpha_k|,$$

we see that f attains its maximum in the interval $[-M_1, M_1]$. Further, according to Bernstein's inequality, we have

$$\|f'\|_{\mathcal{B}_{\pi,0}^\infty} \leq \pi\|f\|_{\mathcal{B}_{\pi,0}^\infty} \leq \max_{|k|\leq N}|\alpha_k| \left(5\pi + 4 + 4\log(N)\right),$$

where we used (4) in the second inequality. Thus, it follows that

$$|f(t_1) - f(t_2)| \leq |t_1 - t_2| \|f'\|_{\mathcal{B}_{\pi,0}^\infty} \leq |t_1 - t_2| \max_{|k|\leq N}|\alpha_k| \left(5\pi + 4 + 4\log(N)\right)$$

for all $t_1, t_2 \in [-M_1, M_1]$. This inequality and the fact that $f(t)$ is computable for all $t \in [-M_1, M_1] \cap \mathbb{R}_c$ show that

$$\|f\|_{\mathcal{B}_{\pi,0}^\infty} = \max_{t\in[-M_1,M_1]} |f(t)|$$

is computable.

A function in $f \in \mathcal{B}_{\pi,0}^\infty$ is computable in $\mathcal{B}_{\pi,0}^\infty$ if there exist a computable sequence of elementary computable functions $\{f_n\}_{n\in\mathbb{N}}$ and a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$, such that for all $N \in \mathbb{N}$ we have $\|f - f_n\|_{\mathcal{B}_{\pi,0}^\infty} \leq 2^{-N}$ for all $n \geq \xi(N)$. By $\mathcal{CB}_{\pi,0}^\infty$ we denote the set of all functions in $\mathcal{B}_{\pi,0}^\infty$ that are computable in $\mathcal{B}_{\pi,0}^\infty$. Note that $\mathcal{CB}_{\pi,0}^\infty$ has a linear structure. The set $\mathcal{CB}_{\pi,0}^\infty$ is the "effective closure" in $\mathcal{B}_{\pi,0}^\infty$ of the set of elementary computable functions. Hence, we can approximate any function $f \in \mathcal{CB}_{\pi,0}^\infty$ by an elementary computable function where we have an "effective" control of the approximation error.

In other words, for every prescribed approximation error $\epsilon > 0$ we can compute an index $n_0 = \xi(\lceil -\log_2(\epsilon) \rceil)$ such that the approximation error $\|f - f_n\|_{\mathcal{B}_\pi^p}$ is less than or equal to ϵ for all $n \geq n_0$. This behavior is illustrated in Fig. 3.

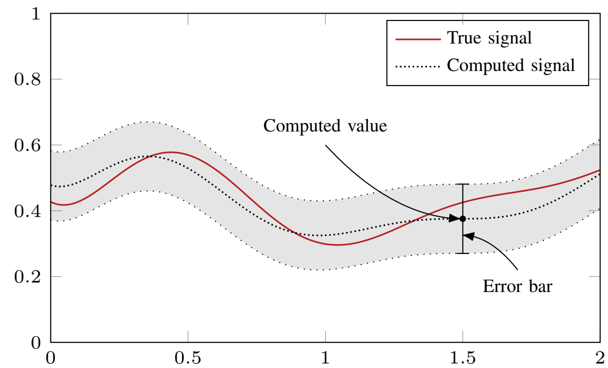

Fig. 3. For a computable signal we can always determine an error bar and then can be sure that the true value lies within the specified error range.

Due to the inequality

$$\left| \|f\|_{\mathcal{B}_{\pi,0}^\infty} - \|f_n\|_{\mathcal{B}_{\pi,0}^\infty} \right| \leq \|f - f_n\|_{\mathcal{B}_{\pi,0}^\infty},$$

it follows immediately that the norm $\|f\|_{\mathcal{B}_{\pi,0}^\infty}$, i.e., the maximum of f, is computable for all $f \in \mathcal{CB}_{\pi,0}^\infty$. See also [34, pp. 40].

In order that the above definition of a computable function in $\mathcal{B}_{\pi,0}^\infty$ makes sense, it is necessary that each $f \in \mathcal{B}_{\pi,0}^\infty$ can be approximated in a classical sense by a linear combination of shifted sinc-functions. This is assured by the next fact, the proof of which will be given in Appendix A.

Fact 1. *Let $f \in \mathcal{B}_{\pi,0}^\infty$. For every $\epsilon > 0$ there exist an $N \in \mathbb{N}$ and numbers $\{c_k\}_{k=-N}^N$ such that*

$$\left\| f - \sum_{k=-N}^{N} c_k \frac{\sin(\pi(t-k))}{\pi(t-k)} \right\|_{\mathcal{B}_{\pi,0}^\infty} < \epsilon.$$

A set $A \subset \mathbb{N}$ is called recursively enumerable if $A = \emptyset$ or A is the range of a recursive function. A set $A \subset \mathbb{N}$ is called recursive if both A and $\mathbb{N} \setminus A$ are recursively enumerable.

IV. Downsampling for Bandlimited Signals

Let $f \in \mathcal{PW}_{2\pi}^2$ be a bandlimited signal with bandwidth 2π and finite energy. Then f is completely determined by its samples $\{f(k/2)\}_{k\in\mathbb{Z}}$. Removing every second sample, i.e., keeping only the samples $\{x_k^{\text{down}}\}_{k\in\mathbb{Z}} = \{f(k)\}_{k\in\mathbb{Z}}$, corresponds to a downsampling factor of two. The continuous-time signal f_π that corresponds to the downsampled discrete-time signal $\{x_k^{\text{down}}\}_{k\in\mathbb{Z}}$ is given by

$$f_\pi(t) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}, \quad t \in \mathbb{R}. \quad (5)$$

The series in (5) converges in the L^2-norm, as well as uniformly on the real axis, and we have $f_\pi \in \mathcal{PW}_\pi^2 \subset \mathcal{PW}_{2\pi}^2$. Hence, for the signal space $\mathcal{PW}_{2\pi}^2$, downsampling and bandlimited interpolation of the downsampled signal are well-defined.

For $N \in \mathbb{N}$, let

$$(S_N f)(t) = \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}, \quad t \in \mathbb{R}$$

denote the finite Shannon sampling series. For every signal f that is computable in $\mathcal{PW}^2_{2\pi}$, $S_N f$ is computable in $\mathcal{PW}^2_{2\pi}$, because $S_N f$ is the finite sum of computable functions. Further, for fixed $t \in \mathbb{R}_c$, the computable sequence $\{(S_N f)(t)\}_{N \in \mathbb{N}}$ of computable numbers converges effectively to $f(t)$, because

$$|f(t) - (S_N f)(t)| \le \|f - S_N f\|_{\mathcal{PW}^2_{2\pi}}.$$

Hence for $f \in \mathcal{PW}^2_{2\pi}$ and each $t \in \mathbb{R}_c$, the bounded bandlimited interpolation $f_\pi(t)$ is computable.

However, downsampling is also often used for other, larger signal spaces, like $\mathcal{B}^\infty_{2\pi,0}$ or $\mathcal{B}^\infty_{2\pi}$, both of which are important for example in communications. In the present paper we study downsampling for the space $\mathcal{B}^\infty_{2\pi,0}$. A signal $f \in \mathcal{B}^\infty_{2\pi,0}$ is uniquely determined by its samples $f(k/2)$, $k \in \mathbb{Z}$, and for all $T > 0$ we have

$$\lim_{N\to\infty} \max_{[-T,T]} \left| f(t) - \sum_{k=-N}^{N} f\left(\frac{k}{2}\right) \frac{\sin(2\pi(t-\frac{k}{2}))}{2\pi(t-\frac{k}{2})} \right| = 0,$$

i.e., the Shannon sampling series converges locally uniformly to the signal f [40]. Further, we have $\{f(k/2)\}_{k\in\mathbb{Z}} \in c_0$. Clearly, the downsampled discrete-time signal also satisfies $\{f(k)\}_{k\in\mathbb{Z}} \in c_0$. However, the question is whether the bounded bandlimited interpolation $f_\pi \in \mathcal{B}^\infty_\pi$ exists, and if yes, if it can be computed.

It is well-known that there exist sequences that do not possess a bounded bandlimited interpolation. For example, for the sequence

$$x_k = \begin{cases} 0, & k \le 0, \\ \frac{(-1)^k}{\log(1+k)}, & k \ge 1, \end{cases}$$

there exists no signal $f_\pi \in \mathcal{B}^\infty_\pi$ with $f_\pi(k) = x_k$ for all $k \in \mathbb{Z}$ [41]. Note that the situation that is analyzed in the present paper is more complicated. Here, the sequence is not freely chosen, but obtained by downsampling a bounded bandlimited signal. In fact, the signal that we will construct later is a bandpass signal with arbitrarily small effective bandwidth.

V. Existence of the Bounded Bandlimited Interpolation

In the following two theorems the signal

$$\gamma_\delta(t) = e^{i\pi t} g_\delta(t), \quad t \in \mathbb{R}, \tag{6}$$

with

$$g_\delta(t) = \frac{1}{\pi} \int_0^{\delta\pi} \frac{\sin(\omega t)}{\omega \log(\frac{\pi}{\omega})} \, d\omega, \quad t \in \mathbb{R},$$

will play a central role. $\delta \in (0,1)$ is a parameter that specifies the bandwidth of the signal. The signal $g_{1/2}$ is visualized in Fig. 4.

Our first theorem shows that the bounded bandlimited interpolation of the downsampled signal does not always exist.

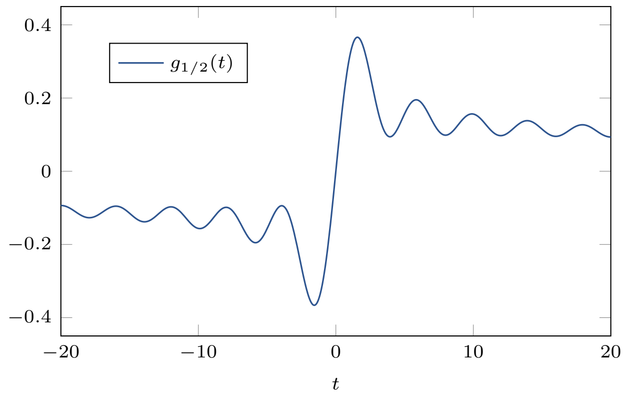

Fig. 4. Plot of the signal $g_\delta(t)$ for $\delta = 1/2$.

Theorem 1. *Let $\delta \in (0,1)$, and let $\gamma_\delta \in \mathcal{B}^\infty_{(1+\delta)\pi,0}$ be the signal defined in (6). Then there exists no $f_\pi \in \mathcal{B}^\infty_\pi$ such that $f_\pi(k) = \gamma_\delta(k)$ for all $k \in \mathbb{Z}$. That is, there exists no bounded bandlimited interpolation for the downsampled sequence $\{\gamma_\delta(k)\}_{k\in\mathbb{Z}}$.*

We postpone all proofs until Section VIII.

As the next theorem shows, for the downsampled sequence $\{\gamma_\delta(k)\}_{k\in\mathbb{Z}}$, the Shannon sampling series diverges for all $t \in \mathbb{R} \setminus \mathbb{Z}$. Moreover, the divergence even holds in a distributional setting.

Theorem 2. *Let $\delta \in (0,1)$, and let $\gamma_\delta \in \mathcal{B}^\infty_{(1+\delta)\pi,0}$ be the signal defined in (6). Then, for all $t \in \mathbb{R} \setminus \mathbb{Z}$, we have*

$$\lim_{N\to\infty} \left| \sum_{k=-N}^{N} \gamma_\delta(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = \infty.$$

Further, there exists a $\phi_1 \in C^\infty_0[0,1]$ such that

$$\lim_{N\to\infty} \left| \int_{-\infty}^{\infty} \sum_{k=-N}^{N} \gamma_\delta(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \phi_1(t) \, dt \right| = \infty,$$

i.e., the series diverges in $\mathcal{D}'$.

In order to illustrate the divergence behavior observed in Theorem 2, the partial sums of the Shannon sampling series

$$(S_N \gamma_\delta)(t) = \sum_{k=-N}^{N} \gamma_\delta(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}$$

are plotted in Fig. 5 for $\delta = 1/2$ and $N = 5, 40, 320$. While the values of the partial sums $(S_N \gamma_\delta)(t)$ are fixed on the integer grid $\mathbb{Z}$, the increase for $t \in \mathbb{R} \setminus \mathbb{Z}$ is clearly visible.

Remark 1. The signal γ_δ has a remarkably simple structure. It is not constructed as an infinite series, but defined as a simple integral expression.

Remark 2. γ_δ is a bandpass signal that is created by modulating the lowpass signal g_δ. Since the spectrum of the lowpass signal g_δ is concentrated on $[-\delta\pi, \delta\pi]$, g_δ is completely determined by its samples $\{g_\delta(k/\delta)\}_{k\in\mathbb{Z}}$. Further, the effective bandwidth of the bandpass signal γ_δ is $2\delta\pi$.

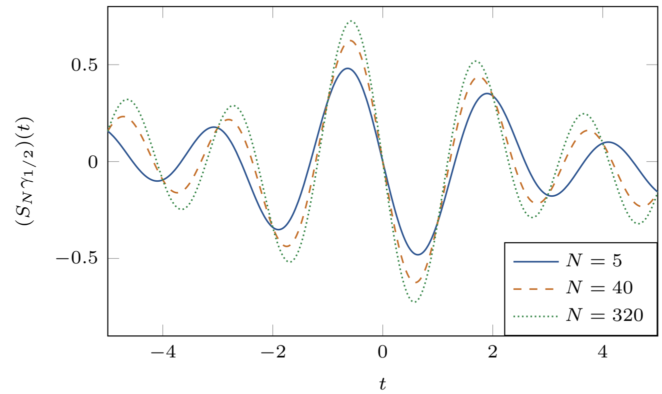

Fig. 5. Plot of the sums $(S_N\gamma_\delta)(t)$ for $\delta = 1/2$ and $N = 5, 40, 320$.

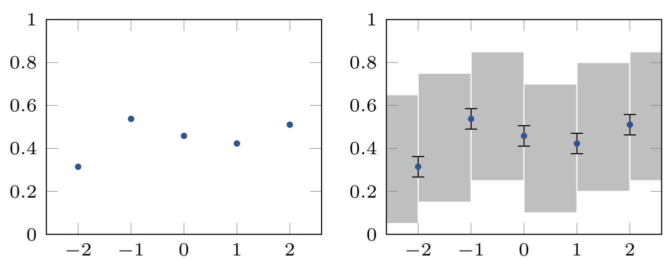

Fig. 6. Illustration of the non-computability of the bandlimited interpolation. In the left panel we see the sequence for which we seek the bandlimited interpolation. In the right panel we see that the values of the bandlimited interpolation are computable on the integer grid $\mathbb{Z}$ (the true value lies within the error bar), but not in between (indicated by the gray rectangles).

VI. Computability of the Bounded Bandlimited Interpolation

In this section we consider the same scenario as before, but now from a computability perspective.

In practical applications, a digital computer is often used to perform the signal processing operations. With respect to the correspondence between discrete-time and continuous-time signals that was discussed in the introduction, the question arises if this correspondence still holds from a computational point of view.

If we sample a computable continuous-time signal $f \in \mathcal{CB}^\infty_{2\pi,0}$, then the sequence of sampling points $\{f(k/2)\}_{k\in\mathbb{Z}}$ is a computable sequence in c_0. The same is true for the downsampled sequence $\{x_k^{\text{down}}\}_{k\in\mathbb{Z}} = \{f(k)\}_{k\in\mathbb{Z}}$. Hence, this direction can be performed without problems. However, the opposite direction, i.e., computing the bounded bandlimited interpolation f_π can be problematic.

In Theorem 1 we have seen that there exist signals, such that for the downsampled discrete-time signal there exists no bounded bandlimited interpolation. Clearly, if the bounded bandlimited interpolation does not exist, we cannot compute it. Hence, in the following we consider only those signals for which the bounded bandlimited interpolation exists. But even if the bounded bandlimited interpolation exists, it is not guaranteed that it is computable. The following theorem is our main result about the computability of the bounded bandlimited interpolation.

Theorem 3. *There exists a computable signal $f \in \mathcal{CB}^\infty_{2\pi,0}$ such that the bandlimited interpolation f_π exists in $\mathcal{B}^\infty_{\pi,0}$ and we have $f_\pi(t) \notin \mathbb{C}_c$ for all $t \in \mathbb{R}_c \setminus \mathbb{Z}$.*

The signal f_π in Theorem 3 is not computable because it does not even satisfy the minimal requirement that computable numbers are mapped into computable numbers. Only for $k \in \mathbb{Z}$ do we have this property, because $f_\pi(k) = f(k)$, $k \in \mathbb{Z}$. Hence, our result shows that f_π is neither Turing, nor Markov, nor Banach–Mazur computable.

Remark 3. Note that Theorem 3 is not only an abstract existence result. In fact, the computable signal f from Theorem 3 is constructed in the proof of the theorem, and given there in eq. (22).

Remark 4. The full proof of Theorem 3 will be given in Section VIII. Nevertheless, we will next give a preview of the construction of the signal f. f is constructed such that f is computable in $\mathcal{B}^\infty_{2\pi,0}$, i.e., can be effectively approximated by a finite sampling series. It follows that the sequence of samples $\{f(k)\}_{k\in\mathbb{Z}}$ is computable in c_0, i.e., $\{f(k)\}_{k\in\mathbb{Z}}$ can be approximated arbitrarily well with a discrete-time signal that only has finitely many non-zero values. This signal is given by the even coefficients of the finite sampling series. However, although the samples $\{f(k)\}_{k\in\mathbb{Z}}$ are computable, their oscillation is too strong, and therefore the bandlimited interpolation $f_\pi(t)$ is not computable for any $t \in \mathbb{R}_c$.

From an approximation point of view, we can interpret the non-computability of f_π in $\mathcal{B}^\infty_{\pi,0}$ as follows. Since f_π is not computable in $\mathcal{B}^\infty_{\pi,0}$, there exists no computable sequence of elementary computable functions $\{f_n\}_{n\in\mathbb{N}}$ that effectively approximates f_π. This means we cannot effectively control the approximation error $\|f_\pi - f_n\|_{\mathcal{B}^\infty_{\pi,0}}$. The result in Theorem 3 is even stronger. Since $f_\pi(t)$ is not computable for all $t \in \mathbb{R}_c \setminus \mathbb{Z}$, not only the approximation with elementary computable functions that employ the sinc function has to fail, but in fact the approximation with any other sequence of computable functions also. This can be seen as follows. Assume that there exists a computable sequence $\{f_n\}_{n\in\mathbb{N}}$ of computable functions, which are not necessarily elementary computable functions, that converges effectively to f_π in $\mathcal{B}^\infty_{\pi,0}$. Since

$$|f_\pi(t) - f_n(t)| \le \|f_\pi - f_n\|_{\mathcal{B}^\infty_{\pi,0}},$$

it follows that, for all $t \in \mathbb{R}_c$, the computable sequence $\{f_n(t)\}_{k\in\mathbb{Z}}$ of computable numbers converges effectively to $f_\pi(t)$. Hence, we have $f_\pi(t) \in \mathbb{C}_c$ for all $t \in \mathbb{R}_c$, which is a contradiction to Theorem 3. The preceding calculation shows that non-computability of $f_\pi(t)$ for $t \in \mathbb{R}_c \setminus \mathbb{Z}$ immediately excludes the possibility that f_π can be effectively approximated by any computable sequence of computable functions in $\mathcal{B}^\infty_{\pi,0}$, even if we allow more general functions than elementary computable functions for the approximation.

Remark 5. From Fact 1 we know that f_π can be approximated by a sequence of finite Shannon sampling series. For every $M \in \mathbb{N}$ there exist an $N(M)$ and numbers $\{c_k(M)\}_{k=-N(M)}^{N(M)}$

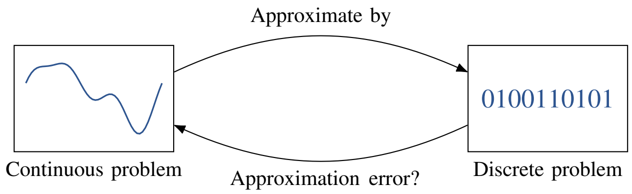

Fig. 7. Connection of the continuous problem and the discrete problem.

such that

$$\left\| f_\pi - \sum_{k=-N(M)}^{N(M)} c_k(M) \frac{\sin(\pi(\cdot - k))}{\pi(\cdot - k)} \right\|_{\mathcal{B}_{\pi,0}^\infty} < \frac{1}{2^M}. \quad (7)$$

However, the pair of functions $N(M)$ and $\{c_k(M)\}_{k=-N(M)}^{N(M)}$ is not Turing computable. Thus, these parameters in (7) cannot be determined algorithmically.

A direct consequence of Theorem 3 is the following corollary.

Corollary 1. *There exists a computable sequence $\{x_k\}_{k\in\mathbb{Z}} \in c_0$ such that the bandlimited interpolation f_π exists in $\mathcal{B}_{\pi,0}^\infty$ and we have $f_\pi(t) \notin \mathbb{C}_c$ for all $t \in \mathbb{R}_c \setminus \mathbb{Z}$.*

Corollary 1 is illustrated in Fig. 6, where it is symbolically shown that $f_\pi(t)$ cannot be approximated for $t \in \mathbb{R}_c \setminus \mathbb{Z}$ with an effective control of the approximation error.

VII. Discussion of the Results

Before we continue with giving the proofs in Section VIII, we resume the discussion started in the introduction about the relevance of the results for signal processing.

In signal processing we face the situation that we have to deal with continuous-time and discrete-time signals. While many real world processes are described mathematically by continuous models, e.g., partial differential equations such as Maxwell's equations or the convection-diffusion equation, and thus involve continuous-time signals, most of the actual signal processing is performed in the digital domain, using digital computers that process discrete-time signals.

A usual approach is to approximate the continuous problem by a discrete problem, which is then solved on a digital computer. But only if we can control the approximation error, does the discrete problem give us useful information about the original continuous problem. Whenever we use a digital computer to solve a continuous problem, we have to ensure that the result is meaningful, which requires controlling the approximation error, as illustrated in Fig. 7.

In his landmark paper "On computable numbers, with an application to the Entscheidungsproblem" from 1936, Alan Turing studied computable real numbers and how they can be approximated by a Turing machine [31]. This is exactly the situation that we discussed above. The real number, which is a continuous quantity, is approximated by a sequence of discrete objects, where the approximation error can be controlled. With this paper, Turing laid the foundation of computer science, employing a model of computation that is close to the hardware that is still used in digital signal processing.

In signal processing, sampling is used to convert a bandlimited continuous-time signal with infinite duration into a sequence of numbers, and bandlimited interpolation to convert a sequence of numbers back into a bandlimited signal [4]–[6], [8], [42]. The Shannon's sampling theorem is the theoretical basis for the coupling of the analog and the discrete domain [2], [13], [14]. The conversion of a continuous-time signal into a discrete-time signal can always be easily done, i.e., any computable bandlimited signals leads to a computable discrete-time signal after sampling. However, the conversion of a computable discrete-time signal into a computable bandlimited signal can be problematic.

Even if the bandlimited interpolation of a discrete-time signal exists mathematically, it might not be possible to compute it algorithmically on a digital computer, because the approximation error cannot be effectively controlled. This is the statement of Theorem 3 and Corollary 1, and is illustrated in Fig. 6.

Our results show that there can be problems in the computation of the bandlimited interpolation on a digital hardware platform. So far, the usual theoretical analyses in signal processing have not treated this kind of problem when designing algorithms. As already discussed, it has implicitly been assumed that digital computers can approximate any continuous problem and that the approximation error decreases with increasing computational power. This is not always the case. Our results show that it is not sufficient to develop an algorithm for the approximation of a continuous problem, rather it is necessary to analyze the approximation error and to specify a stopping criterion that can guarantee any desired approximation error. This is a completely new design problem that has not been treated so far.

Theoretically, it is possible to implement the Shannon sampling series in analog hardware. However, it is completely unclear what such an analog implementation would look like.

In [43] it has been shown, using the Fourier transform as an example, that analog implementations can potentially have advantages compared to digital implementations. More specifically, in [43] it was proved that there exist computable bandlimited signals with finite energy, the Fourier transform of which is smooth but not computable. Turing machines provide a model for the perfect digital machine. Yet, the Fourier transform of such signals cannot be computed on a Turing machine. On the other hand, there is the theory of Fourier optics, where, using an idealized analog machine, it is theoretically possible to compute the Fourier transform [44], [45]. This shows that there is a fundamental difference between the two computing models, given by digital Turing machines and analog Fourier optics. Certainly, the theoretical possibility of implementing the Fourier transform by using Fourier optics makes no statement about how well such an implementation would be realized under real conditions. The answer to this question also depends on the actual state of the hardware development. Further studies in this direction are necessary.

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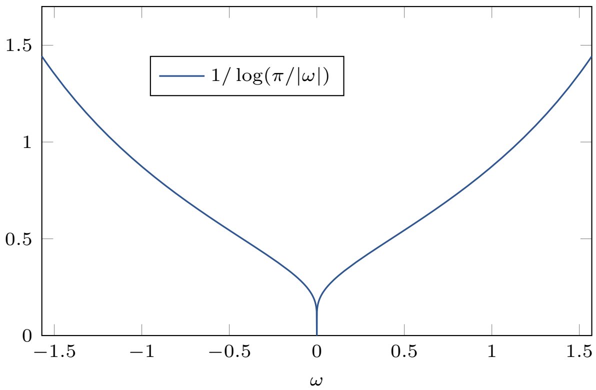

Fig. 8. Plot of the function $1/\log(\pi/|\omega|)$.

VIII. Proofs

In this section we give all postponed proofs. We start with proving several properties of

$$g_\delta(t) = \frac{1}{\pi}\int_0^{\delta\pi} \frac{\sin(\omega t)}{\omega \log(\frac{\pi}{\omega})}\,\mathrm{d}\omega, \quad t\in\mathbb{R},$$

which is illustrated in Fig. 4. In particular, we will show that g_δ, $\delta\in(0,1)$, is a bounded bandlimited signal that vanishes at infinity.

Lemma 1. *Let $\delta\in(0,1)$. Then we have $g_\delta\in\mathcal{B}^\infty_{\delta\pi,0}$. Further, g_δ satisfies $g_\delta(0)=0$ and $g_\delta(t)=-g_\delta(-t)$ for all $t\in\mathbb{R}$.*

Proof. Differentiating g_δ we obtain

$$\begin{aligned} g_\delta'(t) &= \frac{1}{\pi}\int_0^{\delta\pi}\frac{\cos(\omega t)}{\log(\frac{\pi}{\omega})}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\delta\pi}^{\delta\pi}\frac{1}{\log(\frac{\pi}{|\omega|})}\,\mathrm{e}^{i\omega t}\,\mathrm{d}\omega, \end{aligned}$$

where the interchange of integration and differentiation is allowed because $1/\log(\pi/|\omega|)$, and hence

$$\frac{\sin(\omega t)}{\omega\log(\frac{\pi}{\omega})}$$

as well as

$$\frac{\cos(\omega t)}{\log(\frac{\pi}{\omega})}$$

are continuous functions on $[-\delta\pi,\delta\pi]$ (see Fig. 8). We further have

$$\frac{1}{2\pi}\int_{-\delta\pi}^{\delta\pi}\frac{1}{(\log(\frac{\pi}{|\omega|}))^2}\,\mathrm{d}\omega<\infty.$$

Thus, we see that $g_\delta'\in\mathcal{PW}^2_{\delta\pi}$. This implies that g_δ is a function of exponential type at most $\delta\pi$, and consequently that g_δ is bandlimited with bandwidth $\delta\pi$.

Next, we prove that $\lim_{t\to\infty} g_\delta(t)=0$. Let

$$u_t(\omega)=\frac{1}{\pi}\int_0^{\omega}\frac{\sin(\omega_1 t)}{\omega_1}\,\mathrm{d}\omega_1=\frac{1}{\pi}\int_0^{t\omega}\frac{\sin(\omega_2)}{\omega_2}\,\mathrm{d}\omega_2.$$

Using integration by parts, we obtain

$$\pi g_\delta(t)=\frac{1}{\log(\frac{1}{\delta})}u_t(\delta\pi)-\int_0^{\delta\pi}\frac{1}{(\log(\frac{\pi}{\omega}))^2}\frac{1}{\omega}u_t(\omega)\,\mathrm{d}\omega.$$

Since

$$\int_0^{\delta\pi}\frac{1}{(\log(\frac{\pi}{\omega}))^2}\frac{1}{\omega}\,\mathrm{d}\omega=\frac{1}{\log(\frac{1}{\delta})},$$

we see that the function

$$q(\omega)=\frac{1}{(\log(\frac{\pi}{\omega}))^2}\frac{1}{\omega}$$

satisfies $q\in L^1[0,\delta\pi]$. Further, we have $|u_t(\omega)|\le C_1$ for all $t>0$ and $\omega\in[0,\delta\pi]$. Application of Lebesgue's dominated convergence theorem gives

$$\begin{aligned}\lim_{t\to\infty}\int_0^{\delta\pi}\frac{1}{(\log(\frac{\pi}{\omega}))^2}\frac{1}{\omega}u_t(\omega)\,\mathrm{d}\omega &= \int_0^{\delta\pi}\frac{1}{2(\log(\frac{\pi}{\omega}))^2}\frac{1}{\omega}\,\mathrm{d}\omega \\ &= \frac{1}{2\log(\frac{1}{\delta})},\end{aligned}$$

because

$$\lim_{t\to\infty}u_t(\omega)=\frac{\pi}{2}$$

for $\omega>0$ [46, p. 58, Eq. (8.4)]. Hence, it follows that

$$\begin{aligned}&\lim_{t\to\infty}\left(\frac{1}{\log(\frac{1}{\delta})}u_t(\delta\pi)-\int_0^{\delta\pi}\frac{1}{(\log(\frac{\pi}{\omega}))^2}\frac{1}{\omega}u_t(\omega)\,\mathrm{d}\omega\right)\\ &=\frac{\pi}{2}\left(\frac{1}{\log(\frac{1}{\delta})}-\frac{1}{\log(\frac{1}{\delta})}\right)\\ &=0,\end{aligned}$$

which in turn implies that $\lim_{t\to\infty} g_\delta(t)=0$. The properties $g_\delta(0)=0$, and $g_\delta(t)=-g_\delta(-t)$ for all $t\in\mathbb{R}$ follow directly from the definition of g_δ. ☐

For the proofs of our theorems we need several auxiliary results that we state next.

We start with two facts about the local behavior of the Shannon sampling series for signals in $\mathcal{B}^\infty_{\pi,0}$ and $\mathcal{B}^\infty_\pi$, respectively [40, Theorem 1].

Fact 2. *Let $T>0$ be arbitrary. Then, for all $f\in\mathcal{B}^\infty_{\pi,0}$, we have*

$$\lim_{N\to\infty}\max_{t\in[-T,T]}\left|f(t)-\sum_{k=-N}^{N}f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}\right|=0.$$

In particular, Fact 2 shows that every signal $f\in\mathcal{B}^\infty_{\pi,0}$ is uniquely determined by its samples $f(k)$, $k\in\mathbb{Z}$. If $f\in\mathcal{B}^\infty_{\pi,0}$ with $f(k)=0$ for all $k\in\mathbb{Z}$, then it follows that $f\equiv 0$.

The next fact is a statement about the local behavior of the Shannon sampling series for signals in $\mathcal{B}^\infty_\pi$ [40, Theorem 1].

Fact 3. *For all $T>0$ there exists a constant $C_2(T)$ such that, for all $f\in\mathcal{B}^\infty_\pi$ and all $N\in\mathbb{N}$, we have*

$$\max_{t\in[-T,T]}\left|\sum_{k=-N}^{N}f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}\right|\le C_2(T)\|f\|_{\mathcal{B}^\infty_\pi}.$$

Second, we need three facts about the properties of

$$\sum_{k=1}^{\infty}\frac{\sin(k\omega)}{k}, \tag{8}$$

which is the Fourier series of the function

$$u(\omega) = \begin{cases} \frac{1}{2}(\pi - \omega), & 0 < \omega < 2\pi, \\ 0, & \omega = 0 \text{ or } \omega = 2\pi. \end{cases}$$

First, the Fourier series (8) converges pointwise to $u(\omega)$ for all $0 \le \omega \le 2\pi$ [46, p. 5, Eq. 2.8].

Fact 4. *For all $0 \le \omega \le 2\pi$, we have*

$$\sum_{k=1}^{\infty} \frac{\sin(k\omega)}{k} = u(\omega).$$

Second, on all closed intervals, excluding the jump discontinuities, we even have uniform convergence [46, p. 4, Theorem 2.6].

Fact 5. *For all $\gamma > 0$, we have*

$$\lim_{N\to\infty} \max_{\omega\in[\gamma,2\pi-\gamma]} \left| u(\omega) - \sum_{k=1}^{N} \frac{\sin(k\omega)}{k} \right| = 0.$$

Third, the partial sums of (8) are strictly positive on the interval $(0, \pi)$ [46, p. 62, Theorem 9.4].

Fact 6. *For all $N \ge 1$ and all $\omega \in (0, \pi)$, we have*

$$\sum_{k=1}^{N} \frac{\sin(k\omega)}{k} > 0.$$

Now we are in the position to prove Theorem 1.

Proof of Theorem 1. Let $\delta \in (0, 1)$ be arbitrary but fixed and let γ_δ be the signal defined in (6). Then we have $\gamma_\delta \in \mathcal{B}^\infty_{(1+\delta)\pi,0}$. We further have

$$\gamma_\delta(k) = \mathrm{e}^{ik\pi}\, g_\delta(k) = (-1)^k g_\delta(k),$$

$k \in \mathbb{Z}$. Thus, for $t \in \mathbb{R} \setminus \mathbb{Z}$, we obtain

$$\begin{aligned}(S_N\gamma_\delta)(t) &= \sum_{k=-N}^{N} (-1)^k g_\delta(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \\ &= \frac{\sin(\pi t)}{\pi} \sum_{k=-N}^{N} \frac{g_\delta(k)}{t-k} \\ &= \frac{\sin(\pi t)}{\pi} \sum_{k=1}^{N} g_\delta(k) \left(\frac{1}{t-k} - \frac{1}{t+k} \right),\end{aligned}$$

where we used that $\sin(\pi(t-k)) = (-1)^k \sin(\pi t)$. It follows that

$$\begin{aligned}&(S_N\gamma_\delta)(t) + \frac{\sin(\pi t)}{\pi} \sum_{k=1}^{N} \frac{2g_\delta(k)}{k} \\ &\quad= \frac{\sin(\pi t)}{\pi} \sum_{k=1}^{N} g_\delta(k) \left(\frac{t}{(t-k)k} + \frac{t}{(t+k)k} \right).\end{aligned}$$

For $t \in [1/4, 3/4]$, we therefore have

$$\begin{aligned}&\left| (S_N\gamma_\delta)(t) + \frac{\sin(\pi t)}{\pi} \sum_{k=1}^{N} \frac{2g_\delta(k)}{k} \right| \\ &\le \frac{1}{\pi} \sum_{k=1}^{N} g_\delta(k) \left| \frac{t}{(t-k)k} \right| + \frac{1}{\pi} \sum_{k=1}^{N} g_\delta(k) \left| \frac{t}{(t+k)k} \right| \\ &\le \frac{\|g_\delta\|_{\mathcal{B}^\infty_{\delta\pi,0}}}{\pi} \left(\frac{3}{4} \sum_{k=1}^{N} \frac{1}{(k-\frac{3}{4})k} + \frac{3}{4} \sum_{k=1}^{N} \frac{1}{(\frac{1}{4}+k)k} \right) \\ &< \frac{3\|g_\delta\|_{\mathcal{B}^\infty_{\delta\pi,0}}}{4\pi} \left(4 + \sum_{k=2}^{N} \frac{1}{(k-\frac{3}{4})k} + \sum_{k=1}^{N} \frac{1}{k^2} \right).\end{aligned}$$

Since

$$\sum_{k=2}^{N} \frac{1}{(k-\frac{3}{4})k} \le \sum_{k=2}^{N} \frac{1}{(k-1)^2} = \sum_{k=1}^{N-1} \frac{1}{k^2},$$

we obtain

$$\begin{aligned}&\left| (S_N\gamma_\delta)(t) + \frac{\sin(\pi t)}{\pi} \sum_{k=1}^{N} \frac{2g_\delta(k)}{k} \right| \\ &\qquad< \frac{3\|g_\delta\|_{\mathcal{B}^\infty_{\delta\pi,0}}}{4\pi} \left(4 + 2\sum_{k=1}^{\infty} \frac{1}{k^2} \right) \\ &\qquad= \frac{12+\pi^2}{4\pi} \|g_\delta\|_{\mathcal{B}^\infty_{\delta\pi,0}}. \end{aligned} \tag{9}$$

It follows that

$$(S_N\gamma_\delta)(t) \le -\frac{2\sin(\frac{\pi}{4})}{\pi} \sum_{k=1}^{N} \frac{g_\delta(k)}{k} + \frac{12+\pi^2}{4\pi} \|g_\delta\|_{\mathcal{B}^\infty_{\delta\pi,0}} \tag{10}$$

for all $t \in [1/4, 3/4]$, where we used the fact that $\sin(\pi t) \ge \sin(\pi/4)$ for all $t \in [1/4, 3/4]$ in the last equality.

Let $0 < \gamma < \delta\pi$ be arbitrary. Then, due to Fact 6, we have

$$\begin{aligned}\sum_{k=1}^{N} \frac{g_\delta(k)}{k} &= \frac{1}{\pi} \int_0^{\delta\pi} \frac{1}{\omega \log(\frac{\pi}{\omega})} \sum_{k=1}^{N} \frac{\sin(\omega k)}{k} \,\mathrm{d}\omega \\ &\ge \frac{1}{\pi} \int_\gamma^{\delta\pi} \frac{1}{\omega \log(\frac{\pi}{\omega})} \sum_{k=1}^{N} \frac{\sin(\omega k)}{k} \,\mathrm{d}\omega.\end{aligned}$$

Since, according to Fact 5, the series

$$\sum_{k=1}^{\infty} \frac{\sin(\omega k)}{k} \,\mathrm{d}\omega$$

converges uniformly on $[\gamma, \delta\pi]$ to $(\pi - \omega)/2$, we obtain

$$\begin{aligned}\lim_{N\to\infty} \sum_{k=1}^{N} \frac{g_\delta(k)}{k} &\ge \frac{1}{\pi} \int_\gamma^{\delta\pi} \frac{1}{\omega \log(\frac{\pi}{\omega})} \frac{1}{2}(\pi - \omega) \,\mathrm{d}\omega \\ &= \frac{1}{2} \int_\gamma^{\delta\pi} \frac{1}{\omega \log(\frac{\pi}{\omega})} \,\mathrm{d}\omega - \frac{1}{2\pi} \int_\gamma^{\delta\pi} \frac{1}{\log(\frac{\pi}{\omega})} \,\mathrm{d}\omega \\ &> \frac{1}{2} \int_\gamma^{\delta\pi} \frac{1}{\omega \log(\frac{\pi}{\omega})} \,\mathrm{d}\omega - \frac{\delta}{2\log(\frac{1}{\delta})}.\end{aligned}$$

For the integral we have

$$\frac{1}{2}\int_{\gamma}^{\delta\pi} \frac{1}{\omega \log(\frac{\pi}{\omega})}\, \mathrm{d}\omega = -\frac{1}{2}\int_{\log(\frac{\pi}{\gamma})}^{\log(\frac{1}{\delta})} \frac{1}{u}\, \mathrm{d}u = \frac{1}{2}\log\left(\frac{\log(\frac{\pi}{\gamma})}{\log(\frac{1}{\delta})}\right),$$

which gives

$$\lim_{N\to\infty}\sum_{k=1}^{N}\frac{g_\delta(k)}{k} > \frac{1}{2}\log\left(\frac{\log(\frac{\pi}{\gamma})}{\log(\frac{1}{\delta})}\right) - \frac{\delta}{2\log(\frac{1}{\delta})}$$

for all γ with $0<\gamma<\delta\pi$. Taking the limit $\gamma\to 0$ shows that

$$\lim_{N\to\infty}\sum_{k=1}^{N}\frac{g_\delta(k)}{k} = \infty. \tag{11}$$

Combining (10) and (11), we see that

$$\lim_{N\to\infty}(S_N\gamma_\delta)(t) = -\infty \tag{12}$$

for all $t\in[1/4,3/4]$.

Assume that there exists a signal $f_\pi \in \mathcal{B}_\pi^\infty$ with $f_\pi(k) = \gamma_\delta(k)$, $k\in\mathbb{Z}$. Then, according to Fact 3, we have

$$\begin{aligned}\max_{t\in[-T,T]}\left|\sum_{k=-N}^{N}\gamma_\delta(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}\right| &= \max_{t\in[-T,T]}\left|\sum_{k=-N}^{N}f_\pi(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}\right| \\ &\le C_2(T)\|f\|_{\mathcal{B}_\pi^\infty}\end{aligned}$$

for all $N\in\mathbb{N}$ and $T>0$. This is a contradiction (12). Thus, there exists no signal $f_\pi\in\mathcal{B}_\pi^\infty$ with $f_\pi(k)=\gamma_\delta(k)$, $k\in\mathbb{Z}$. □

Proof of Theorem 2. From the proof of Theorem 1 we already know that for the signal $\gamma_\delta\in\mathcal{B}_{(1+\delta)\pi,0}^\infty$, $\delta\in(0,1)$, we have

$$\lim_{N\to\infty}(S_N\gamma_\delta)(1/2) = -\infty. \tag{13}$$

Let $t_1\in\mathbb{R}\setminus\mathbb{Z}$. We have

$$\begin{aligned}\left|\frac{(S_N\gamma_\delta)(\frac{1}{2})}{\sin(\frac{\pi}{2})} - \frac{(S_N\gamma_\delta)(t_1)}{\sin(\pi t_1)}\right| &\le \frac{1}{\pi}\sum_{k=-N}^{N}\frac{|g_\delta(k)||t_1-\frac{1}{2}|}{|\frac{1}{2}-k||t_1-k|} \\ &\le \frac{\|g_\delta\|_{\mathcal{B}_{\delta\pi,0}^\infty}}{\pi}\sum_{k=-N}^{N}\frac{|t_1-\frac{1}{2}|}{|\frac{1}{2}-k||t_1-k|} \\ &\le \|g_\delta\|_{\mathcal{B}_{\delta\pi,0}^\infty}C_3(t_1),\end{aligned}$$

where $C_3(t_1)$ is a positive constant that depends on t_1 but not on N. The upper bound $C_3(t_1)$ for the sum can be obtained by a calculation that is similar to the calculation that led to (9). It follows that

$$\left|\frac{(S_N\gamma_\delta)(t_1)}{\sin(\pi t_1)}\right| \ge \left|\frac{(S_N\gamma_\delta)(\frac{1}{2})}{\sin(\frac{\pi}{2})}\right| - \|g_\delta\|_{\mathcal{B}_{\delta\pi,0}^\infty}C_3(t_1),$$

which, using (13), implies

$$\lim_{N\to\infty}|(S_N\gamma_\delta)(t_1)| = \infty. \tag{14}$$

This proves the first assertion.

Let ϕ_1 be a function in $C_0^\infty[0,1]$ with $\phi_1(t)\ge 0$ for all $t\in\mathbb{R}$ and

$$\phi_1(t) = \begin{cases}1, & \frac{2}{5}\le t\le\frac{3}{5},\\ 0, & t\in\mathbb{R}\setminus(\frac{1}{4},\frac{3}{4}).\end{cases}$$

From (10) we know that

$$(S_N\gamma_\delta)(t) \le -\frac{2\sin(\frac{\pi}{4})}{\pi}\sum_{k=1}^{N}\frac{g_\delta(k)}{k} + \frac{12+\pi^2}{4\pi}\|g_\delta\|_{\mathcal{B}_{\delta\pi,0}^\infty}$$

for all $t\in[1/4,3/4]$. It follows that

$$\begin{aligned}\int_{-\infty}^{\infty}(S_N\gamma_\delta)(t)\phi_1(t)\,\mathrm{d}t &= \int_{1/4}^{3/4}(S_N\gamma_\delta)(t)\phi_1(t)\,\mathrm{d}t \\ &\le -\int_{1/4}^{3/4}\frac{2\sin(\frac{\pi}{4})}{\pi}\sum_{k=1}^{N}\frac{g_\delta(k)}{k}\phi_1(t)\,\mathrm{d}t \\ &\quad + \int_{1/4}^{3/4}\frac{12+\pi^2}{4\pi}\|g_\delta\|_{\mathcal{B}_{\delta\pi,0}^\infty}\phi_1(t)\,\mathrm{d}t \\ &\le -\frac{2}{5}\frac{\sin(\frac{\pi}{4})}{\pi}\sum_{k=1}^{N}\frac{g_\delta(k)}{k} + \frac{12+\pi^2}{4\pi}\|g_\delta\|_{\mathcal{B}_{\delta\pi,0}^\infty}\|\phi_1\|_{L^1(\mathbb{R})},\end{aligned}$$

and, using (11), that

$$\lim_{N\to\infty}\int_{-\infty}^{\infty}(S_N\gamma_\delta)(t)\phi_1(t)\,\mathrm{d}t = -\infty. \tag{15}$$

This completes the proof of Theorem 2. □

For the proof of Theorem 3 we need several additional lemmas.

Lemma 2. *Let $f_1,f_2\in\mathcal{B}_{\pi,0}^\infty$ with $f_1(k)=f_2(k)$ for all $k\in\mathbb{Z}$. Then we have $f_1\equiv f_2$.*

For a proof of Lemma 2, see, for example, [47, p. 155, Theorem 1].

A sampling series that is valid for a larger signal space than the Shannon sampling series is the Valiron sampling series [7, p. 12], which is sometimes called Tschakaloff's series [8, p. 60]. This series provides a valid sampling representation for signals in $\mathcal{B}_\pi^\infty$.

Lemma 3. *For all $f\in\mathcal{B}_\pi^\infty$, we have*

$$f(t) = f(0)\frac{\sin(\pi t)}{\pi t} + f'(0)\frac{\sin(\pi t)}{\pi} + t\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{f(k)}{k}\frac{\sin(\pi(t-k))}{\pi(t-k)}, \quad t\in\mathbb{R}.$$

For fixed $t\in\mathbb{R}$, the series converges absolutely.

We also need the fact that for computable functions $f\in\mathcal{CB}_{\pi,0}^\infty$ and fixed $t\in\mathbb{R}_c$, the third term in the Valiron sampling series is a computable number.

Lemma 4. *Let $f\in\mathcal{CB}_{\pi,0}^\infty$ and $t\in\mathbb{R}_c$. Then we have*

$$t\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{f(k)}{k}\frac{\sin(\pi(t-k))}{\pi(t-k)} \in \mathbb{R}_c.$$

Proof. Let $f \in \mathcal{CB}^\infty_{\pi,0}$ be arbitrary but fixed. We use the abbreviations

$$B(t) = t \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{f(k)}{k} \frac{\sin(\pi(t-k))}{\pi(t-k)}$$

and

$$B_N(t) = t \sum_{\substack{k=-N \\ k\neq 0}}^{N} \frac{f(k)}{k} \frac{\sin(\pi(t-k))}{\pi(t-k)}.$$

For $t \in \mathbb{R}$ and $N \in \mathbb{N}$ with $N > |t|$, we have

$$\begin{aligned} |B(t) - B_N(t)| &\leq \frac{|t|}{\pi} \sum_{|k|>N} \left| \frac{f(k)}{k} \frac{\sin(\pi(t-k))}{t-k} \right| \\ &\leq \frac{|t|}{\pi} \|f\|_{\mathcal{B}^\infty_{\pi,0}} \sum_{|k|>N} \frac{1}{|k(t-k)|} \end{aligned}$$

and

$$\begin{aligned} \sum_{|k|>N} \frac{1}{|k(t-k)|} &= \sum_{k=-\infty}^{-N-1} \frac{1}{|k(t-k)|} + \sum_{k=N+1}^{\infty} \frac{1}{|k(t-k)|} \\ &= \sum_{k=N+1}^{\infty} \frac{1}{k(t+k)} + \frac{1}{k(k-t)} \\ &= \sum_{k=N+1}^{\infty} \frac{2}{k^2 - t^2} \\ &\leq 2 \sum_{k=N+1}^{\infty} \frac{1}{k^2}. \end{aligned}$$

Since

$$\sum_{k=N+1}^{\infty} \frac{1}{k^2} < \sum_{k=N+1}^{\infty} \int_{k-1}^{k} \frac{1}{\tau^2}\, \mathrm{d}\tau = \int_N^\infty \frac{1}{\tau^2}\, \mathrm{d}\tau = \frac{1}{N},$$

it follows that

$$|B(t) - B_N(t)| < \frac{2|t|}{\pi N} \|f\|_{\mathcal{B}^\infty_{\pi,0}} \qquad (16)$$

for all $t \in \mathbb{R}$ and all $N \in \mathbb{N}$ with $N > |t|$. For all $t \in \mathbb{R}_c$, the right-hand side of (16) is a computable number, and thus we see that the computable sequence $\{B_N(t)\}_{N\in N}$ converges effectively to $B(t)$. This implies that $B(t) \in \mathbb{R}_c$ for all $t \in \mathbb{R}_c$. □

Lemma 5. *For all $t \in \mathbb{R}$, we have*

$$\sum_{k=-\infty}^{\infty} \left(\frac{\sin(\pi(t-k))}{\pi(t-k)} \right)^2 = 1.$$

Proof. Let $t \in \mathbb{R}$ be arbitrary but fixed. According to Parseval's theorem we have

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \left(\frac{\sin(\pi(t-k))}{\pi(t-k)} \right)^2 &= \int_{-\infty}^{\infty} \left(\frac{\sin(\pi(t-\tau))}{\pi(t-\tau)} \right)^2 \mathrm{d}\tau \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\mathrm{e}^{i\omega t}|^2 \,\mathrm{d}\omega = 1, \end{aligned}$$

which completes the proof. □

Lemma 6. *Let $A \subset \mathbb{N}$ be a recursively enumerable nonrecursive set, and $\phi_A \colon \mathbb{N} \to \mathbb{N}$ a recursive enumeration of the elements of A, where ϕ_A is a one-to-one function, i.e., for every element $k \in A$ there exists exactly one $N_k \in \mathbb{N}$ with $\phi_A(N_k) = k$. Then the number*

$$\sum_{N=1}^{\infty} \frac{1}{2^{\phi_A(N)}}$$

is not computable.

We will prove Lemma 6 in Section X.

Now we are in the position to prove Theorem 3.

Proof of Theorem 3. In this proof we will construct a signal f that is computable in $\mathcal{B}^\infty_{2\pi,0}$. It is easy to see that the sequence of samples $\{f(k)\}_{k\in\mathbb{Z}}$ is computable in c_0. Since $f \in \mathcal{CB}^\infty_{2\pi,0}$, we know from Fact 1 and the fact that f is computable, that for every approximation error $1/2^M$, f can be approximated effectively in the $\mathcal{B}^\infty_{2\pi,0}$-norm by a finite series

$$\sum_{k=-N(M)}^{N(M)} c_k(M) \frac{\sin(\pi(t-k))}{\pi(t-k)},$$

such that the approximation error is less than $1/2^M$. This implies that the sequence of samples $\{f(k)\}_{k\in\mathbb{Z}}$ can be effectively approximated in the c_0-norm with a discrete-time signal that has only finitely many non-zero values. Although the samples $\{f(k)\}_{k\in\mathbb{Z}}$ are computable, we will see later that their oscillation is too strong so that the bandlimited interpolation $f_\pi(t)$ in $\mathcal{B}^\infty_{\pi,0}$ is not computable for any $t \in \mathbb{R}_c \setminus \mathbb{Z}$.

Let $N \in \mathbb{N}$ be arbitrary but fixed, and let

$$p_N(t) = -\sum_{k=1}^{N} (-1)^k \left(\frac{\sin(\pi(t-k))}{\pi(t-k)} \right)^2, \quad t \in \mathbb{R}.$$

As a finite sum of computable functions, p_N is computable in $\mathcal{B}^\infty_{2\pi,0}$. Hence, we see that $p_N \in \mathcal{CB}^\infty_{2\pi,0}$. Further, we have

$$|p_N(t)| \leq \sum_{k=1}^{N} \left(\frac{\sin(\pi(t-k))}{\pi(t-k)} \right)^2 \leq 1, \qquad (17)$$

where we used Lemma 5 in the second inequality. Since $|p_N(k)| = 1$ for all $1 \leq k \leq N$, it follows that

$$\|p_N\|_{\mathcal{B}^\infty_{2\pi,0}} = 1.$$

Let

$$\begin{aligned} p_{N,\pi}(t) &= \sum_{k=1}^{N} p_N(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \\ &= -\frac{\sin(\pi t)}{\pi} \sum_{k=1}^{N} \frac{1}{t-k}. \end{aligned} \qquad (18)$$

Then, we have

$$p_{N,\pi}\left(\frac{1}{2}\right) = \frac{1}{\pi} \sum_{k=1}^{N} \frac{1}{k - \frac{1}{2}}.$$

Since

$$\frac{1}{k-\frac{1}{2}} > \int_k^{k+1} \frac{1}{\tau - \frac{1}{2}}\, \mathrm{d}\tau, \quad k \geq 1,$$

it follows that

$$p_{N,\pi}\left(\frac{1}{2}\right) = \frac{1}{\pi}\int_1^{N+1} \frac{1}{\tau - \frac{1}{2}}\, \mathrm{d}\tau > \frac{1}{\pi}\log(2N+1). \quad (19)$$

Further, for $t \in \mathbb{R}\setminus\mathbb{Z}$, we have

$$\begin{aligned}|p_{N,\pi}(t)| &\leq \sum_{k=1}^{N}\left|\frac{\sin(\pi(t-k))}{\pi(t-k)}\right| \\ &< 2 + \frac{1}{\pi}\sum_{k=1}^{k_1(t)}\frac{1}{t-k} + \frac{1}{\pi}\sum_{k=k_2(t)}^{N}\frac{1}{k-t} \\ &< 2 + \frac{1}{\pi}\sum_{k=1}^{k_1(t)}\frac{1}{k_1(t)+1-k} + \frac{1}{\pi}\sum_{k=k_2(t)}^{N}\frac{1}{k-k_2(t)+1} \\ &= 2 + \frac{1}{\pi}\sum_{k=1}^{k_1(t)}\frac{1}{k} + \frac{1}{\pi}\sum_{k=1}^{N-k_2(t)+1}\frac{1}{k} \\ &\leq 2 + \frac{2}{\pi}\sum_{k=1}^{N}\frac{1}{k} \\ &< 2 + \frac{2}{\pi} + \frac{2}{\pi}\log(N), \end{aligned} \quad (20)$$

where $k_1(t)$ is the largest natural number that is smaller than or equal to N and satisfies $k_1(t)+1 < t$. Further, $k_2(t)$ is the smallest natural number such that $k_2(t)-1 > t$. If $k_2(t) > N$ then the above sums involving $k_2(t)$ are the empty sums. We also used the inequality

$$\begin{aligned}\sum_{k=1}^{N}\frac{1}{k} &< 1 + \sum_{k=2}^{N}\int_{k-1}^{k}\frac{1}{\tau}\,\mathrm{d}\tau \\ &= 1 + \int_1^N \frac{1}{\tau}\,\mathrm{d}\tau \\ &= 1 + \log(N)\end{aligned}$$

in the last line. This shows that

$$\|p_{N,\pi}\|_{\mathcal{B}_{\pi,0}^\infty} \leq 2 + \frac{2}{\pi} + \frac{2}{\pi}\log(N).$$

Further, we see from (18) that

$$p_{(N+1),\pi}\left(\frac{1}{2}\right) > p_{N,\pi}\left(\frac{1}{2}\right). \quad (21)$$

Let

$$g_N(t) = \frac{1}{p_{N,\pi}(\frac{1}{2})}p_N(t), \quad t \in \mathbb{R}.$$

Then we have

$$\begin{aligned}\|g_N\|_{\mathcal{B}_{2\pi,0}^\infty} &= \frac{1}{|p_{N,\pi}(\frac{1}{2})|}\|p_N\|_{\mathcal{B}_{2\pi,0}^\infty} \\ &< \frac{\pi}{\log(2N+1)},\end{aligned}$$

where we used (17) and (19) in the last inequality.

Let $A \subset \mathbb{N}$ be an arbitrary recursively enumerable nonrecursive set, and let $\phi_A\colon \mathbb{N} \to \mathbb{N}$ be an enumeration of the elements of A, where ϕ_A is a one-to-one function, i.e., for every element $k \in A$ there exists exactly one $N_k \in \mathbb{N}$ with $\phi_A(N_k) = k$. We consider the function

$$f(t) = \sum_{N=1}^{\infty}\frac{1}{2^{\phi_A(N)}}g_N(t), \quad t \in \mathbb{R}. \quad (22)$$

Since

$$\begin{aligned}\sum_{N=1}^{\infty}\left\|\frac{1}{2^{\phi_A(N)}}g_N\right\|_{\mathcal{B}_{2\pi,0}^\infty} &\leq \sum_{N=1}^{\infty}\frac{1}{2^{\phi_A(N)}}\|g_N\|_{\mathcal{B}_{2\pi,0}^\infty} \\ &< \sum_{N=1}^{\infty}\frac{1}{2^N}\frac{\pi}{\log(2N+1)} \\ &< \pi,\end{aligned}$$

it follows that the series in (22) is absolutely convergent and that $f \in \mathcal{B}_{2\pi,0}^\infty$. For $M \in \mathbb{N}$, we have

$$\begin{aligned}\left\|f - \sum_{N=1}^{M}\frac{1}{2^{\phi_A(N)}}g_N\right\|_{\mathcal{B}_{2\pi,0}^\infty} &\leq \sum_{N=M+1}^{\infty}\frac{1}{2^{\phi_A(N)}}\|g_N\|_{\mathcal{B}_{2\pi,0}^\infty} \\ &< \frac{1}{p_{(M+1),\pi}\left(\frac{1}{2}\right)}\sum_{N=M+1}^{\infty}\frac{1}{2^N} \\ &\leq \frac{1}{p_{(M+1),\pi}\left(\frac{1}{2}\right)} \\ &\leq \frac{\pi}{\log(2M+3)},\end{aligned}$$

where we used (21) in the second inequality and (19) in the last inequality. This shows that the computable sequence

$$\left\{\sum_{N=1}^{M}\frac{1}{2^{\phi_A(N)}}g_N\right\}_{M=1}^{\infty}$$

converges effectively to f. Hence, f is computable in $\mathcal{B}_{2\pi,0}^\infty$. Let

$$g_{N,\pi}(t) = \frac{1}{p_{N,\pi}(\frac{1}{2})}p_{N,\pi}(t), \quad t \in \mathbb{R},$$

and

$$q_*(t) = \sum_{N=1}^{\infty}\frac{1}{2^{\phi_A(N)}}g_{N,\pi}(t), \quad t \in \mathbb{R}. \quad (23)$$

For $N \in \mathbb{N}$ we have

$$\begin{aligned}\|g_{N,\pi}\|_{\mathcal{B}_{\pi,0}^\infty} &= \frac{1}{p_{N,\pi}\left(\frac{1}{2}\right)}\|p_{N,\pi}\|_{\mathcal{B}_{\pi,0}^\infty} \\ &< \frac{\pi}{\log(2N+1)}\left(2 + \frac{2}{\pi} + \frac{2}{\pi}\log(N)\right) \\ &\leq \frac{2(1+\pi)}{\log(3)},\end{aligned}$$

and it follows that

$$\|q_*\|_{\mathcal{B}_{\pi,0}^\infty} \leq \sum_{N=1}^{\infty}\frac{1}{2^{\phi_A(N)}}\|g_{N,\pi}\|_{\mathcal{B}_{\pi,0}^\infty} < \frac{2(1+\pi)}{\log(3)}.$$

Hence, we see that the series in (23) converges absolutely. Further, according to Lemma 2, we have $f_\pi = q_*$, because $f(k) = q_*(k)$ for all $k \in \mathbb{Z}$, and f_π as well as q_* are in $\mathcal{B}_{\pi,0}^\infty$. Since, according to Lemma 6

$$\sum_{N=1}^{\infty}\frac{1}{2^{\phi_A(N)}}$$

is not computable, it follows that

$$f_\pi\left(\frac{1}{2}\right) = q_*\left(\frac{1}{2}\right) = \sum_{N=1}^{\infty} \frac{1}{2^{\phi_A(N)}} g_{N,\pi}\left(\frac{1}{2}\right) = \sum_{N=1}^{\infty} \frac{1}{2^{\phi_A(N)}}$$

is not computable, i.e., we have $f_\pi(1/2) \notin \mathbb{R}_c$.

Using Lemma 3 we see that

$$f_\pi(t) = \underbrace{f_\pi(0)\frac{\sin(\pi t)}{\pi t}}_{=:B_1(t)} + f_\pi'(0)\frac{\sin(\pi t)}{\pi} + \underbrace{t \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{f_\pi(k)}{k}\frac{\sin(\pi(t-k))}{\pi(t-k)}}_{=:B_2(t)}, \quad t \in \mathbb{R}. \tag{24}$$

We analyze the three summands on the right-hand side of (24). We have $B_1(t) \in \mathbb{R}_c$ for all $t \in \mathbb{R}_c$, because $f_\pi(0) = f(0) \in \mathbb{R}_c$. Further, we have $B_2(t) \in \mathbb{R}_c$ for all $t \in \mathbb{R}_c$, due to Lemma 4. For $t = 1/2$ we know that $f_\pi(1/2) \notin \mathbb{R}_c$. Thus, the left-hand side and, consequently, the right-hand side of (24) are not computable for $t = 1/2$. It follows that $f_\pi'(0) \notin \mathbb{R}_c$. This implies that

$$f_\pi'(0)\frac{\sin(\pi t)}{\pi},$$

and, consequently, $f_\pi(t)$ is not computable for all $t \in \mathbb{R}_c \setminus \mathbb{Z}$. □

IX. Size of the Set of Problematic Signals

It would be interesting to know if the problematic behavior that was observed in Theorems 1 and 2 only exists for very few signals or if it is the prevailing behavior that occurs for almost all signals. The next theorem gives an answer in terms of Baire categories.

We review some of the definitions. A subset $\mathcal{M}$ of a Banach space $\mathcal{X}$ is said to be nowhere dense in $\mathcal{X}$ if the interior of the closure of $\mathcal{M}$ is empty. $\mathcal{M}$ is said to be of first category (or meager) if $\mathcal{M}$ is the countable union of sets, each of which is nowhere dense in $\mathcal{X}$. $\mathcal{M}$ is said to be of second category (or nonmeager) if is not of first category. The complement of a set of first category is called a residual set. Topologically, sets of first category may be considered "small". Accordingly, residual sets, being the complements of sets of first category, can be considered "large". In a complete metric space, any residual set is dense and a set of second category, due to Baire's theorem [48].

Theorem 4. *Let $\delta \in (0,1)$. The set of all signals $f \in \mathcal{B}^\infty_{(1+\delta)\pi,0}$, for which there exists no $f_\pi \in \mathcal{B}^\infty_\pi$ with $f_\pi(k) = f(k)$ for all $k \in \mathbb{Z}$, is a residual set in $\mathcal{B}^\infty_{(1+\delta)\pi,0}$.*

Theorem 5. *Let $\delta \in (0,1)$. The set of all signals $f \in \mathcal{B}^\infty_{(1+\delta)\pi,0}$, for which we have*

$$\limsup_{N\to\infty} \left| \sum_{k=-N}^{N} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = \infty$$

for all $t \in \mathbb{R}\setminus\mathbb{Z}$, is a residual set in $\mathcal{B}^\infty_{(1+\delta)\pi,0}$. Further, the set of all signals $f \in \mathcal{B}^\infty_{(1+\delta)\pi,0}$, for which there exists a $\phi \in C_0^\infty[0,1]$ such that

$$\limsup_{N\to\infty} \left| \int_{-\infty}^{\infty} \sum_{k=-N}^{N} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}\phi(t)\,\mathrm{d}t \right| = \infty,$$

is a residual set in $\mathcal{B}^\infty_{(1+\delta)\pi,0}$.

This shows that "almost all" signals in $\mathcal{B}^\infty_{(1+\delta)\pi,0}$ exhibit the problematic behavior with respect to downsampling.

The previous two theorems were formulated with regard to the bandpass signals $f \in \mathcal{B}^\infty_{(1+\delta)\pi,0}$. It is also possible to make a statement about the set of lowpass signals $g \in \mathcal{B}^\infty_{\delta\pi,0}$ for which the downsampled sequence $\{f(k)\}_{k\in\mathbb{Z}}$, obtained from the continuous-time signal

$$f(t) = \mathrm{e}^{i\pi t}\, g(t), \quad t \in \mathbb{R},$$

has no bounded bandlimited interpolation.

Theorem 6. *Let $\delta \in (0,1)$. The set of all signals $g \in \mathcal{B}^\infty_{\delta\pi,0}$, for which we have*

$$\limsup_{N\to\infty} \left| \sum_{k=-N}^{N} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = \infty$$

for all $t \in \mathbb{R}\setminus\mathbb{Z}$, is a residual set in $\mathcal{B}^\infty_{\delta\pi,0}$. Further, the set of all signals $g \in \mathcal{B}^\infty_{\delta\pi,0}$, for which there exists a $\phi \in C_0^\infty[0,1]$ such that

$$\limsup_{N\to\infty} \left| \int_{-\infty}^{\infty} \sum_{k=-N}^{N} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}\phi(t)\,\mathrm{d}t \right| = \infty,$$

is a residual set in $\mathcal{B}^\infty_{\delta\pi,0}$. In both statements above, $f(t) = \mathrm{e}^{i\pi t}\, g(t)$, $t \in \mathbb{R}$.

Proof of Theorem 4. A look at the proof of Theorem 1 reveals that it is sufficient to prove that there exists a $t \in \mathbb{R}\setminus\mathbb{Z}$ such that the set of signals $f \in \mathcal{B}^\infty_{(1+\delta)\pi,0}$, for which we have $\limsup_{N\to\infty}|(S_N f)(t)| = \infty$, is a residual set. Having shown this, the rest of the proof is done analogously to the proof of Theorem 1.

Next, we prove the above statement. Let $t \in \mathbb{R}\setminus\mathbb{Z}$ be arbitrary but fixed. For $N \in \mathbb{N}$, let

$$\psi_N f = (S_N f)(t) = \sum_{k=-N}^{N} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}.$$

For each $N \in \mathbb{N}$, $\psi_N\colon \mathcal{B}^\infty_{(1+\delta)\pi,0} \to \mathbb{C}$ is a continuous linear functional. Since

$$\|\psi_N\| = \sup_{\substack{f\in\mathcal{B}^\infty_{(1+\delta)\pi,0} \\ \|f\|_{\mathcal{B}^\infty_{(1+\delta)\pi,0}}\leq 1}} |\psi_N f| \geq |\psi_N \gamma_{1/2}| = |(S_N\gamma_{1/2})(t)|,$$

it follows from (14) that $\lim_{N\to\infty}\|\psi_N\| = \infty$. Hence, the Banach–Steinhaus theorem [49, p. 98] implies that the set of signals $f \in \mathcal{B}^\infty_{(1+\delta)\pi,0}$ with

$$\limsup_{N\to\infty}|\psi_N f| = \limsup_{N\to\infty}|(S_N f)(t)| = \infty$$

is a residual set. □

Proof of Theorem 5. Let $\phi_1 \in C_0^\infty[0,1]$ be the function from Theorem 2, and let $t \in \mathbb{R} \setminus \mathbb{Z}$ be arbitrary but fixed. Further, for $N \in \mathbb{N}$, let

$$\Gamma_N f = \int_{-\infty}^{\infty} \left(\sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right) \phi_1(t) \, \mathrm{d}t.$$

For each $N \in \mathbb{N}$, $\Gamma_N \colon \mathcal{B}_{(1+\delta)\pi,0}^\infty \to \mathbb{C}$ is a continuous linear functional. Let γ_δ, $\delta \in (0,1)$, be the function from Theorem 1. Since $\lim_{N\to\infty} |\Gamma_N \gamma_\delta| = \infty$, according to (15), it follows from the Banach–Steinhaus theorem [49, p. 98] that the set of signals $f \in \mathcal{B}_{(1+\delta)\pi,0}^\infty$ for which

$$\limsup_{N\to\infty} |\Gamma_N f| = \limsup_{N\to\infty} \left| \int_{-\infty}^{\infty} (S_N f)(t) \phi_1(t) \, \mathrm{d}t \right| = \infty$$

is a residual set. □

Proof of Theorem 6. Let $t \in \mathbb{R} \setminus \mathbb{Z}$ be arbitrary but fixed. For $N \in \mathbb{N}$, let

$$\tilde{\psi}_N g = \sin(\pi t) \sum_{k=-N}^{N} \frac{g(k)}{\pi(t-k)}.$$

For each $N \in \mathbb{N}$, $\tilde{\psi}_N \colon \mathcal{B}_{\delta\pi,0}^\infty \to \mathbb{C}$ is a continuous linear functional. Since

$$\begin{aligned}(S_N f)(t) &= \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \\ &= \sum_{k=-\infty}^{\infty} (-1)^k g(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \\ &= \sin(\pi t) \sum_{k=-N}^{N} \frac{g(k)}{\pi(t-k)} \\ &= \tilde{\psi}_N g,\end{aligned}$$

we see that

$$\|\tilde{\psi}_N\| = \sup_{\substack{g \in \mathcal{B}_{\delta\pi,0}^\infty \\ \|g\|_{\mathcal{B}_{\delta\pi,0}^\infty} \le 1}} |\tilde{\psi}_N g| \ge |\tilde{\psi}_N g_{1/2}| = |(S_N \gamma_{1/2})(t)|,$$

and it follows from (14) that $\lim_{N\to\infty} \|\tilde{\psi}_N\| = \infty$. The Banach–Steinhaus theorem [49, p. 98] shows that the set of signals $g \in \mathcal{B}_{\delta\pi,0}^\infty$ for which $\limsup_{N\to\infty} |\tilde{\psi}_N g| = \infty$ is a residual set. □

X. Construction of a Non-Computable Number

In this section we prove Lemma 6 and give an interpretation of the approximation of a non-computable number in terms of computable Cauchy sequences. A proof of Lemma 6 is given in [34, p. 17] using basic facts about computable sequences. The proof that we will give in the following is different from the proof in [34, p. 17]. It is based on dyadic expansions and illustrates the essential properties of non-computable numbers.

A rational number $x \in (0,1)$ is called dyadic rational if we have $x = m/2^N$ for some $m, N \in \mathbb{N}$. Without loss of generality we can assume that m and 2^N are coprime. Clearly, we always have $m < 2^N$, and m has the representation

$$m = \sum_{l=0}^{L} a_l(m) 2^l,$$

where $a_l(m) \in \{0,1\}$, $0 \le l \le L$, and $L < N$ is the smallest natural number such that $a_l(m) = 0$ for all $l > L$. For dyadic rational numbers we have the representation

$$x = \sum_{n=1}^{\infty} a_n(x) \frac{1}{2^n}$$

with $a_n(x) = 0$ for $n > N$. For every number $x \in (0,1)$ that is not dyadic rational, we have the unique representation

$$x = \sum_{n=1}^{\infty} a_n(x) \frac{1}{2^n}.$$

Let $A \subset \mathbb{N}$ be a recursively enumerable nonrecursive set, and let $\phi_A \colon \mathbb{N} \to \mathbb{N}$ be an enumeration of the elements of A, where ϕ_A is a one-to-one function, i.e., for every element $k \in A$ there exists exactly one $N_k \in \mathbb{N}$ with $\phi_A(N_k) = k$. Further, let

$$x_A = \sum_{n=1}^{\infty} \frac{1}{2^{\phi_A(n)}}.$$

For x_A, which is not dyadic rational, we have

$$a_n(x_A) = \begin{cases} 1, & n \in A, \\ 0, & n \in \mathbb{N} \setminus A. \end{cases}$$

Proof of Lemma 6. We assume that

$$x_A = \sum_{n=1}^{\infty} \frac{1}{2^{\phi_A(n)}} \in \mathbb{R}_c$$

and construct a contradiction. Since x_A is computable, there exist a computable sequence $\{x_n\}_{n\in\mathbb{N}}$ of rational numbers and a recursive function $\phi \colon \mathbb{N} \to \mathbb{N}$ such that $|x_A - x_n| \le 1/2^N$ for all $n \ge \phi(N)$. We have $a_k(x_A) = a_k(x_n)$ for all $1 \le k \le N-1$. This follows from $|2^{N-1} x_A - 2^{N-1} x_n| \le 1/2$ for all $n \ge \phi(N)$, which shows that the integer parts of $2^{N-1} x_A$ and $2^{N-1} x_n$ need to be equal. Let $n = \phi(N)$. Since $x_{\phi(N)}$ is a computable rational number, it follows that the coefficients $a_k(x_{\phi(N)})$, $1 \le k \le N-1$ are computable. This implies for $1 \le k \le N-1$ and all $N \in \mathbb{N}$ that $k \in A$ if $a_k(x_{\phi(N)}) = 1$ and $k \in \mathbb{N} \setminus A$ if $a_k(x_{\phi(N)}) = 0$. Hence, A is a recursive set, which is a contradiction to our assumption. □

Let $\{x_n\}_{n\in\mathbb{N}}$ be an arbitrary computable sequence of computable numbers with $\lim_{n\to\infty} x_n = x_A$. A computable sequence of computable numbers $\{x_n\}_{n\in\mathbb{N}}$ is called a computable Cauchy sequence if there exists a computable function ϕ such that for all $N \in \mathbb{N}$ we have

$$|x_{M_1} - x_{M_2}| < \frac{1}{2^N} \tag{25}$$

for all $M_1, M_2 \ge \phi(N)$. Since x_A is not computable, $\{x_n\}_{n\in\mathbb{N}}$ cannot be a computable Cauchy sequence. We even have for every computable function ϕ: Every recursively enumerable infinite set $I \subset \mathbb{N}$ contains an infinite subset $\underline{I} \subset I$ such that for all $N \in \underline{I}$ there exist two numbers, $M_1(N), M_2(N) \in \mathbb{N}$ with $M_1(N) \ge \phi(N)$ and $M_2(N) \ge \phi(N)$, such that

$$|x_{M_1(N)} - x_{M_2(N)}| \ge \frac{1}{2^N}.$$

That is, for every recursively enumerable set $I \subset \mathbb{N}$, the condition (25) is violated infinitely often. As a consequence, if we implement any computable stopping algorithm, the stopping time condition (25) is violated infinitely often.

XI. Conclusion

In classical signal processing books, the theoretical treatment of the two operations, downsampling and bandlimited interpolation, is not given special attention, despite their high importance in applications. There are no studies of the analytical properties of downsampling for bandlimited signals that vanish at infinity. The usual narrative is that the bandlimited interpolation always exists [25, p. 52 and p. 162] and [26, p. 144]. That this cannot be true for arbitrary signals has been demonstrated in [41], where a sequence in c_0 was constructed that possesses no bounded bandlimited interpolation.

In the present paper we went much further and studied the existence of the bandlimited interpolation for sequences that are created by downsampling a discrete-time signal that has been generated by sampling bandlimited signals. By proving that this bandlimited interpolation does not exist in general, we have shown that downsampling needs to be treated carefully when considering more general signal spaces than $\mathcal{PW}_\sigma^2$, the space of bandlimited signals with finite energy. Further, we analyzed the algorithmic computability of the bandlimited interpolation and proved that even when the bandlimited interpolation exists mathematically, it cannot always be computed on a digital computer, because the approximation error cannot be controlled.

To the best of our knowledge, there have been no rigorous studies of this problem so far, and our result is the first in this direction.

Appendix

Proof of Fact 1. Let $f \in \mathcal{B}_{\pi,0}^\infty$ be arbitrary. For $\delta > 0$, we consider the function

$$g_\delta(t) = f((1-\delta)t)\frac{\sin(\delta\pi t)}{\delta\pi t}.$$

We have $g_\delta \in \mathcal{PW}_\pi^2$. Further, we have

$$\begin{aligned}
&|f(t) - g_\delta(t)| \\
&= \left| f(t) - f(t)\frac{\sin(\delta\pi t)}{\delta\pi t} + f(t)\frac{\sin(\delta\pi t)}{\delta\pi t} - f((1-\delta)t)\frac{\sin(\delta\pi t)}{\delta\pi t} \right| \\
&\le |f(t)| \left| 1 - \frac{\sin(\delta\pi t)}{\delta\pi t} \right| + |f(t) - f((1-\delta)t)| \left| \frac{\sin(\delta\pi t)}{\delta\pi t} \right| \\
&\le |f(t)| \left| 1 - \frac{\sin(\delta\pi t)}{\delta\pi t} \right| + |f(t) - f((1-\delta)t)|. \qquad (26)
\end{aligned}$$

Let $\epsilon > 0$ be arbitrary but fixed. According to the Riemann–Lebesgue lemma, there exists a $T_1 = T_1(\epsilon)$ such that

$$|f(t)| < \frac{\epsilon}{8} \qquad (27)$$

for all $|t| \ge T_1$. It follows that

$$|f(t)| \left| 1 - \frac{\sin(\delta\pi t)}{\delta\pi t} \right| \le 2|f(t)| < \frac{\epsilon}{4}$$

for all $|t| \ge T_1$. Further, there exists a $\delta_0 = \delta_0(\epsilon)$ such that

$$\max_{|t|\le T_1} \left| 1 - \frac{\sin(\delta\pi t)}{\delta\pi t} \right| < \frac{\epsilon}{4\|f\|_{\mathcal{B}_{\pi,0}^\infty}}$$

for all $0 < \delta \le \delta_0$. Hence, it follows that

$$|f(t)| \left| 1 - \frac{\sin(\delta\pi t)}{\delta\pi t} \right| < \frac{\epsilon}{4} \qquad (28)$$

for all $t \in \mathbb{R}$ and all $0 < \delta \le \delta_0$. For $|t| \ge 2T_1$ and $0 < \delta \le 1/2$ we have

$$\begin{aligned}
|f(t) - f((1-\delta)t)| &\le |f(t)| + |f((1-\delta)t)| \\
&< \frac{\epsilon}{8} + \frac{\epsilon}{8} \\
&= \frac{\epsilon}{4},
\end{aligned}$$

where we used (27) in the last inequality. Further, for $|t| \le 2T_1$ we have

$$\begin{aligned}
|f(t) - f((1-\delta)t)| &\le \|f'\|_{\mathcal{B}_{\pi,0}^\infty} |1 - (1-\delta)t| \\
&= \|f'\|_{\mathcal{B}_{\pi,0}^\infty} \delta|t| \\
&\le \|f'\|_{\mathcal{B}_{\pi,0}^\infty} \delta 2T_1.
\end{aligned}$$

Let $\delta_1 = \delta_1(\epsilon) = \epsilon/(8T_1\|f'\|_{\mathcal{B}_{\pi,0}^\infty})$. Then we have

$$|f(t) - f((1-\delta)t)| < \frac{\epsilon}{4}$$

for $|t| \le 2T_1$ and all $0 < \delta < \delta_1$. It follows that

$$|f(t) - f((1-\delta)t)| < \frac{\epsilon}{4} \qquad (29)$$

for all $t \in \mathbb{R}$ and all $0 < \delta < \min\{1/2, \delta_1\}$. From (26), (28), and (29), we see that

$$|f(t) - g_\delta(t)| < \frac{\epsilon}{2} \qquad (30)$$

for all $t \in \mathbb{R}$ and all $0 < \delta < \min\{1/2, \delta_0, \delta_1\}$.

Let $\hat{\delta} \in (0, \min\{1/2, \delta_0, \delta_1\})$ be arbitrary but fixed. Since $g_{\hat{\delta}} \in \mathcal{PW}_\pi^2$, there exists an $N \in \mathbb{N}$ such that

$$\left\| g_{\hat{\delta}} - \sum_{k=-N}^{N} g_{\hat{\delta}}(k) \frac{\sin(\pi(\,\cdot\, - k))}{\pi(\,\cdot\, - k)} \right\|_{\mathcal{PW}_\pi^2} < \frac{\epsilon}{2}. \qquad (31)$$

Using the abbreviation

$$(S_N g_{\hat{\delta}})(t) = \sum_{k=-N}^{N} g_{\hat{\delta}}(k) \frac{\sin(\pi(t-k))}{\pi(t-k)},$$

we can conclude from (31) that

$$\|g_{\hat{\delta}} - S_N g_{\hat{\delta}}\|_{\mathcal{B}_{\pi,0}^\infty} \le \|g_{\hat{\delta}} - S_N g_{\hat{\delta}}\|_{\mathcal{PW}_\pi^2} < \frac{\epsilon}{2}. \qquad (32)$$

It follows that

$$\begin{aligned}
\|f - S_N g_{\hat{\delta}}\|_{\mathcal{B}_{\pi,0}^\infty} &\le \|f - g_{\hat{\delta}} + g_{\hat{\delta}} - S_N g_{\hat{\delta}}\|_{\mathcal{B}_{\pi,0}^\infty} \\
&\le \|f - g_{\hat{\delta}}\|_{\mathcal{B}_{\pi,0}^\infty} + \|g_{\hat{\delta}} - S_N g_{\hat{\delta}}\|_{\mathcal{B}_{\pi,0}^\infty} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon,
\end{aligned}$$

where we used (30) and (32) in the last inequality. □

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